SUGGESTED SOLUTION TO HOMEWORK 3

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Problem 1. (a) Let p be the Minkowski functional for a convex set $U \subset X$. Show that for $x \neq 0$, $p(x) = 0$ if and only if $x \in tU$ for every $t > 0$.

(a) Show that $p(x) \leq 1$ if $x \in U$, and $p(x) \geq 1$ if $x \notin U$.

Proof. (a) If $0 \neq x \in tU$ for every $t > 0$, by the definition of $p(x)$, for all $t > 0$ such that $x \in tU$ and

$$t > p(x),$$

which implies $p(x) = 0$. If $x \neq 0$ and $p(x) = 0$, then for arbitrary fixed $\varepsilon > 0$, there exists $t' > p(x) = 0$ such that $x \in t'U$ and

$$t' < p(x) + \varepsilon = \varepsilon.$$

Therefore for arbitrary fixed $t > 0$, choosing $\varepsilon \leq t$, by convexity, we have $x \in t'U \subset tU$.

(b) If $x \in U$, then $p(x) \leq 1$ by the definition. If $x \notin U$, suppose $p(x) < 1$, then for arbitrary $\varepsilon > 0$, there exists $t > p(x)$ such that $x \in tU$ and

$$t < p(x) + \varepsilon,$$

therefore by choosing $\varepsilon = 1 - p(x)$, we have $p(x) < t < 1$, then $x \in tU \subset U$ which is a contradiction. Therefore $p(x) \geq 1$ for $x \notin U$. □

Problem 2. Let M be the set of sequences in the real space c_{00} for which the leading nonzero term is positive. Show that the sets M and $-M$ are convex and disjoint, but they cannot be separated by a hyperplane.

Proof. Recall that c_{00} is the set of sequences x such that $x(n) = 0$ for all but finitely many $n \in \mathbb{N}$. We show that M and $-M$ are convex and disjoint. It is clear that M and $-M$ are disjoint. For $x_1, x_2 \in M$, and $0 \leq \lambda \leq 1$, we assume that there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$x_1(n) = x_2(n) = 0,$$

therefore

$$\lambda x_1(n) + (1 - \lambda)x_2(n) = 0,$$

which implies $\lambda x_1 + (1 - \lambda)x_2 \in c_{00}$. Moreover, suppose the leading nonzero terms in x_1 and x_2 are $x_1(n_1) > 0$ and $x_2(n_2) > 0$ respectively, with out loss of generality, we assume $n_1 \leq n_2$, therefore

$$\lambda x_1(n_1) + (1 - \lambda)x_2(n_1) \geq \lambda x_1(n_1) > 0,$$

which implies $\lambda x_1 + (1 - \lambda)x_2 \in M$, therefore M is a convex set. In the similar way, we can prove that $-M$ is also a convex set.

Then suppose to the contrary, that there exist $f \in (c_{00})^*$ and constant $c \in \mathbb{R}$ such that

$$f(x) \geq c > f(y), \quad \forall x \in M, y \in -M.$$

For $i \in \mathbb{N}$, let e_i be defined as

$$e_i(j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Then since $e_i \in M$ and $-e_i \in -M$, therefore

$$f(e_i) \geq c, \quad f(e_i) > -c.$$

If $c = 0$, then for all $i \in \mathbb{N}$,

$$f(e_i) > 0.$$

Taking

$$x_0(k) = \begin{cases} 1, & k = 1, \\ -2\frac{f(e_1)}{f(e_2)}, & k = 2, \\ 0, & k \geq 3, \end{cases}$$

then

$$f(x_0) = -f(e_1) < 0,$$

which is a contradiction since $x_0 \in M$. For $c \neq 0$, without loss of generality, we assume $c > 0$, since $(c_{00})^* = \ell^1$, there exists $f' \in \ell^1$ such that for all $x \in c_{00}$,

$$f(x) = \sum_{i=1}^{\infty} f'(i)x(i),$$

then for each $i \in \mathbb{N}$, by taking $x = e_i$,

$$f'(e_i) \geq c > 0,$$

therefore $f' \notin \ell^1$ which is a contradiction. □

Problem 3. Let $C([0,1])$ be the vector space of continuous functions on $[0,1]$. Define $\delta(x) = x(0)$ for $x \in C([0,1])$.

(a) Show that δ is a bounded linear functional if $C([0,1])$ is endowed with the sup-norm. Find the norm of δ.

(b) Show that δ is an unbounded linear functional if $C([0,1])$ is endowed with the norm

$$\|x\| = \int_0^1 |x(t)|dt.$$

Proof. (a) It is clear that δ is a linear functional. Moreover, for all $x \in C([0,1])$,

$$|\delta(x)| = |x(0)| \leq \|x\|_\infty,$$

which implies δ is bounded and $\|\delta\|_{(C([0,1]))^*} \leq 1$. Choosing $x \equiv \mathbb{1}_{[0,1]}$, then $\|\mathbb{1}_{[0,1]}\|_\infty = 1$, and

$$\delta(\mathbb{1}_{[0,1]}) = 1,$$

which implies $\|\delta\|_{(C([0,1]))^*} = 1$.

(b) Consider for $n \in \mathbb{N}$,

$$x_n(t) = \begin{cases} -2n^2 t + 2n, & t \in [0, \frac{1}{n}], \\ 0, & t \in (\frac{1}{n}, 1]. \end{cases}$$

Then $\|x_n\|_1 = 1$ and $\delta(x_n) = 2n$, therefore δ is unbounded since we can take n arbitrarily large. □

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